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# DUALITY PRINCIPLE OF CONSERVATION LAWS IN DISLOCATION CONTINUUM

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**Abstract**—The conservation laws in a nonlinear elastic dislocation continuum are studied by combining Noether's theorem with the Nonriemannian geometric theory. The distortion tensors are newly defined in the Lagrangian description and in the Eulerian description in terms of vielbein theory, which results in some simplifications in the theoretical derivation of these laws. The first step of the derivation is to build up two basic equations of variational invariance by applying simultaneously Noether's theorem to the integral actions of a dislocation continuum system, which are expressed both in the Lagrangian and in the Eulerian descriptions, respectively. Then, based on these two equations, as the second step, we use six infinitesimal transformations of both independent and dependent variables to obtain two sets of six specific conservation laws. These transformations are: time translation; material and spatial coordinate translations; material and spatial coordinate rotation as well as rotation of distortion tensors in the natural state. Finally, the duality principle is discussed in some detail between these two sets of conservation equations. In particular, a conservation law called the gauge angular momentum conservation law due to free rotation of anholonomic coordinate frame of a material body in the natural state is believed not to have been revealed before.

## 1. INTRODUCTION

Defect continuum mechanics, as an important developing branch of generalized continuum mechanics, aims at establishing a sound theoretical basis for exploring the elastic and nonelastic behavior of materials with imperfections of various kinds, such as voids, inclusions, inhomogeneities, microcracks, and dislocations, as well as disclinations, on the microscopic and macroscopic levels. A great deal of progress in this field has been made during the last three decades. As far as the dislocation continuum theory is concerned, the most significant contributions, originally due to Kondo's work[1, 2], are due to Kröner,[3, 4] and Bilby *et al.*[5, 6]. They found that the geometric structure of a material manifold with dislocations can be closely related to Nonriemannian geometry. Since then, many papers have been published dealing with the theory of a dislocation continuum (for instance, see Ref. [7–11]).

On the other hand, for a practical application, we need to deal not only with the geometric aspects of the theory, but with the dynamic equations, as well as the conservation laws governing the motion and deformation of dislocated media. As we know, strong interest arose in the study of conservation laws for an elastic continuum since Eshelby[12] introduced the concept of the force on an elastic singularity in terms of an energy-momentum tensor. A. G. Herrmann[13] gave brief account of the history of development in the study of conservation laws for an elastic continuum. These studies[14–17] were generally based on Noether's theorem. This theorem can be stated as follows[18]: *If an action integral of a certain field continuum based on a Lagrangian function satisfying Euler equations of motion remains infinitesimally invariant under some small transformations of independent and/or field variables, there must exist some conservation laws for the field corresponding to these transformations and the number of conservation laws is just equal to that of the transformations.* Furthermore, within the framework of the large deformation theory, where the action integral can be represented either in Lagrangian description or in Eulerian description, the duality principle of conservation laws for both descriptions can be established using Noether's theorem[19, 20].

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The purpose of the present article is to continue the study of the conservation laws within the framework of nonlinear elastic dislocation continuum theory by combining Nonriemannian geometry with Noether's theorem. In Section 2, we shall briefly recall some basic formulations for describing the motion and deformation of a material continuum with dislocations. We shall stress the fact that the distortion tensors are newly defined in the Lagrangian description and in the Eulerian description in terms of vielbein theory, which is often used in the theory of relativity and particle physics[21–23]. In fact, introducing vielbeins to express the distortion tensor will result in some simplification in the theoretical derivation of these laws. In Section 3, we shall present a proof of Noether's theorem and then apply it simultaneously to the integral actions of a dislocation continuum system to build up two basic equations of variational invariance. These actions are expressed both in the Lagrangian and in the Eulerian descriptions, respectively. In Section 4, based on these two fundamental equations, two sets of conservation laws are obtained through six specific infinitesimal transformations of both independent and dependent variables. They are: time translation; material and spatial coordinate translations; material and spatial coordinate rotation as well as rotation of distortion tensors in the natural state. In particular, the duality principle is discussed in some detail between these two sets of conservation equations. A conservation law called the "gauge angular momentum" conservation law due to free rotation of anholonomic coordinate frame in the natural state is believed not to have been discussed before.

## 2. GEOMETRIC DESCRIPTION OF MOTION AND DEFORMATION

The motion and deformation of a material body with continuously distributed dislocations can be described by three different states, namely the reference, the deformed, and the natural states, respectively. Hereafter, we shall refer to them as the r-state, the d-state, and the n-state. The position of any material point $P$ in the r-state is determined by the coordinates $x^\mu$ with the base vectors $\mathbf{e}_\mu$ and the metric tensor

$$e_{\mu\nu} = \mathbf{e}_\mu \cdot \mathbf{e}_\nu \tag{2.1}$$

in a Euclidian space $\mathbf{E}_3$, in which the material body is immersed. When the material body is subjected to external forces from the outside, it moves and deforms from the r-state to the d-state. Meanwhile, new dislocations might be created inside the body. Using the two-point tensor method, (see Eringen *et al*[24, 25]) the motion of the body from the r-state to the d-state or from the d-state to the r-state can be determined by the relations

$$y^a = y^a(x, t) \tag{2.2a}$$

or their inverse relations

$$x^\mu = x^\mu(y, t) \tag{2.2b}$$

respectively, where a new coordinate system $y^a$ with the base vectors $\mathbf{e}_a$ and metric tensor

$$e_{ab} = \mathbf{e}_a \cdot \mathbf{e}_b \tag{2.3}$$

is introduced for the position of the point $P$ in the d-state. Therefore, any incremental vector

$$\mathrm{d}\mathbf{R} = \mathbf{e}_\mu \, \mathrm{d}x^\mu \tag{2.4}$$

taken from the r-state between two material points with the coordinates $x^\mu$ and $x^\mu$ +

$dx^\mu$ is transformed into

$$d\mathbf{r} = dy^a \mathbf{e}_a = y^a_\mu \, dx^\mu \, \mathbf{e}_a = \mathbf{h}_\mu \, dx^\mu \tag{2.5}$$

in the r-state, where we introduced the notation

$$y^a_\mu = \partial_\mu y^a, \qquad \mathbf{h}_\mu = y^a_\mu \mathbf{e}_a, \qquad h_{\mu\nu} = \mathbf{h}_\mu \cdot \mathbf{h}_\nu \tag{2.6}$$

to express, respectively, the displacement gradient and the comoving coordinate base vectors and the corresponding metric tensor. Similarly, using the inverse motion (2.1b), we can also write dR in terms of dr as

$$d\mathbf{R} = dx^\mu \mathbf{e}_\mu = x^\mu_a \, dy^a \, \mathbf{e}_\mu = \mathbf{h}_a \, dy^a \tag{2.7}$$

where

$$x^\mu_a = \partial_a y^a, \qquad \mathbf{h}_a = x^\mu_a \mathbf{e}_\mu, \qquad h_{ab} = \mathbf{h}_a \cdot \mathbf{h}_b. \tag{2.8}$$

As known from dislocation continuum theory[3, 4], the n-state of the material body can be reached by cutting a very small volume spanned by three base vectors $\mathbf{h}_\mu$ in the d-state off from its surroundings and releasing it from the constraints of these surroundings. The process of cutting is described in terms of an affine transformation A of the torn small material element, which is called the distortion tensor. As suggested by Sedov[7, 8], using Gibbs' dyadic notation, this tensor can be written as

$$\mathbf{A} = A^a_\mu \mathbf{e}_a \mathbf{h}^\mu \tag{2.9}$$

where $A^a_\mu$ are the components of A and the index $\mu$ is associated with the comoving base vectors $\mathbf{h}_\mu$. Thus, the distortion tensor (2.9) can be seen as a transformation, through which each line element dr in the d-state is mapped into

$$\delta\mathbf{R} = \mathbf{A} \cdot d\mathbf{r} = A^a_\mu \mathbf{e}_a \mathbf{h}_\mu \cdot \mathbf{h}_\nu \, dx^\nu = A^a_\mu \, dx^\mu \, \mathbf{e}_a \tag{2.10}$$

where $\delta\mathbf{R}$ represents a small line element in the n-state.

However, this way to describe the distortion tensor is not unique. As pointed out by Gairola[26], each relaxed volume element can translate and rotate freely in the n-state. Therefore, we may assume that each element is rotated in such a way that all local frames with the base vectors $\mathbf{e}_A$ and the anholonomic coordinates $z_A$ become parallel. Therefore, the local anholonomic frames are just the fragments of the global Cartesian frame. Using the local base vectors $\mathbf{e}_A$, we may express A as

$$\mathbf{A} = \phi_{aA} \mathbf{e}_A \mathbf{e}^a \tag{2.11a}$$

with its inverse

$$\mathbf{A}^{-1} = \phi^a_A \mathbf{e}_A \mathbf{e}_a \tag{2.11b}$$

where $\phi_{aA}$ is called a vielbein in classic field theory[21]. $\phi^a_A$ represents its contravariant form. As pointed out by Duan and Zhang[23], it is more basic and convenient to make use of the vielbein tensor instead of the metric tensor in the study of energy-momentum conservation law in the general theory of relativity. Based on a similar procedure, we introduce here the vielbein tensor (2.11a) as a fundamental dependent variable to describe the conservation laws in a dislocation continuum. Thus, combining (2.10) with (2.11a), we rewrite $\delta\mathbf{R}$ in (2.10) as

$$\delta\mathbf{R} = \delta z_A \mathbf{e}_A = \phi_{aA} \, dy^a \, \mathbf{e}_A \tag{2.12a}$$

or in component forms

$$\delta z_A = \phi_{aA}\, dy^a = A^a_\mu e_{aA}\, dx^\mu. \tag{2.12b}$$

Consequently, $A^a_\mu$ and $\phi_{aA}$ are related to each other by

$$A^a_\mu = e^a_A \phi_{bA} y^b_\mu, \quad \text{or} \quad \phi_{aA} = x^\mu_a A^b_\mu e_{bA} \tag{2.13}$$

where the notation

$$e^a_A = \mathbf{e}^a \cdot \mathbf{e}_A \qquad e_{aA} = \mathbf{e}_a \cdot \mathbf{e}_A \tag{2.14}$$

represent shifters from the $y^a$-coordinate system to the $\delta z_A$-coordinate system.

On the other hand, from the view-point of Lagrangian representation, we may define another affine transformation **B** by which the line element **dR** seen as a counterpart of dr in the r-state is assumed to be mapped into the same line element $\delta\mathbf{R}$ in the n-state, i.e.

$$\delta\mathbf{R} = \mathbf{B}\cdot d\mathbf{R}. \tag{2.15}$$

Therefore, in a similar way as we did above, this mapping **B** can be expressed in the Gibbs' dyadic notation as

$$\mathbf{B} = B^\mu_a \mathbf{e}_\mu \mathbf{h}^a. \tag{2.16}$$

If we introduce another vielbein $\phi_{\mu A}$, **B** is expressed by

$$\mathbf{B} = \phi_{\mu A} \mathbf{e}_A \mathbf{e}^\mu \tag{2.17a}$$

with its inverse

$$\mathbf{B}^{-1} = \phi^\mu_A \mathbf{e}_A \mathbf{e}_\mu. \tag{2.17b}$$

Substituting (2.16) and (2.17) into (2.15) respectively, we may derive the following relations between $B^\mu_a$ and $\phi_{\mu A}$ as

$$B^\mu_a = e^\mu_A \phi_{\nu A} x^\nu_a, \qquad \phi_{\mu A} = y^a_\mu B^\nu_a e_{\nu A} \tag{2.18}$$

and

$$\delta z_A = B^\mu_a e_{\mu A}\, dy^a = \phi_{\mu A}\, dx^\mu \tag{2.19}$$

where

$$e^\mu_A = \mathbf{e}^\mu \cdot \mathbf{e}_A, \qquad e_{\mu A} = \mathbf{e}_\mu \cdot \mathbf{e}_A \tag{2.20}$$

represent the shifters from the $x^\mu$-coordinate system to the $\delta z_A$-coordinate system. Comparing (2.12b) with (2.19), we obtain

$$\phi_{\mu A} = A^b_\mu e_{bA} = y^a_\mu \phi_{aA} = y^a_\mu \mathrm{B}^\nu_a e_{\nu A} \tag{2.21a}$$

and alternatively

$$\phi_{aA} = B^\nu_a e_{\nu A} = x^\mu_a \phi_{\mu A} = x^\mu_a A^b_\mu e_{bA}. \tag{2.21b}$$

Thus, it is seen that as soon as the motion (2.2a) or its inverse (2.2b) is determined, the vielbeins $\phi_{\mu A}$, $\phi_{aA}$ and the torsion tensors $A^a_\mu$ and $B^\mu_a$ are equivalent to each other. However, the index $\mu$ appearing in $A^a_\mu$ and in $\phi_{\mu A}$ has a different meaning: the former is associated with the comoving base vectors $\mathbf{h}_\mu$, but the latter with the fixed base vectors $\mathbf{e}_\mu$. For the index $a$, the same situation holds for $B^\mu_a$ and $\phi_{aA}$ also.

Through the affine transformation $\mathbf{A}$, the comoving base vectors $\mathbf{h}^\mu$ and $\mathbf{h}_\mu$ are transformed into

$$\mathbf{g}_\mu = \mathbf{A}\cdot\mathbf{h}_\mu = \phi_{\mu A}\mathbf{e}_A \quad \text{and} \quad \mathbf{g}^\mu = \mathbf{A}^{-1}\cdot\mathbf{h}^\mu = \phi^\mu_A\mathbf{e}_A \tag{2.22}$$

respectively, and they represent a local curvilinear coordinate frame. The corresponding metric tensor and the affine connection $\Gamma^\lambda_{\mu\nu}$ are defined by

$$g_{\mu\nu} = \mathbf{g}_\mu\cdot\mathbf{g}_\nu \tag{2.23}$$

and

$$\Gamma^\lambda_{\mu\nu} = \mathbf{g}^\lambda \cdot \frac{\partial \mathbf{g}_\mu}{\partial x^\nu} = \phi^\lambda_A \partial_\mu \phi_{\nu A} \tag{2.24}$$

respectively. It is obvious that if the dislocations do exist inside the body, a space $M$ which is equipped with the metric $g_{\mu\nu}$ and the torsion tensor

$$T^\lambda_{\mu\nu} = \Gamma^\lambda_{[\mu\nu]} \tag{2.25}$$

is a non-Euclidian one. We may also prove from (2.13) that the corresponding Riemann-Christoffel tensor $R^\sigma_{\mu\nu\lambda}$ is identically zero, i.e.

$$R_{\mu\nu\lambda}{}^\sigma = 2\partial_{[\mu}\Gamma^\sigma_{\nu]\lambda} - \Gamma^\rho_{\mu\lambda}\Gamma^\sigma_{\nu\rho} + \Gamma^\sigma_{\mu\rho}\Gamma^\rho_{\nu\lambda} \equiv 0. \tag{2.26}$$

In other words, the Nonriemannian space $M$ is flat.

In the study of dislocation continuum theory, it was an important discovery that the torsion tensor defined in (2.25) is just a mathematical version of dislocation density. If using the vielbein $\phi_{\mu A}$, the dislocation density of the second order in Lagrangian representation can be related to the torsion tensor $T^\lambda_{\mu\nu}$ by

$$\alpha^{\mu\nu} = \epsilon^{\mu\lambda\sigma}_{(e)} T^\nu_{\lambda\sigma} = \epsilon^{\mu\lambda\sigma}_{(e)} \phi^\lambda_A \partial_{[\mu}\phi_{\nu]A} \tag{2.27}$$

where $\epsilon^{\mu\lambda\sigma}_{(e)}$ is the permutation symbol divided by $\sqrt{e}$, where $e = \det(e_{\mu\nu})$.

On the other hand, in Eulerian representation where $y^a$ and time $t$ are taken as independent variables, the local comoving base vectors are

$$\mathbf{g}_a = \mathbf{B}\cdot\mathbf{h}_a = \phi_{aA}\mathbf{e}_A \quad \text{and} \quad \mathbf{g}^a = \mathbf{B}_{-1}\cdot\mathbf{h}^a = \phi^a_A\mathbf{e}_A. \tag{2.28}$$

We can also define a non-Euclidian space $\mathcal{M}$ equipped with the metric tensor

$$g_{ab} = \mathbf{g}_a\cdot\mathbf{g}_b = \phi_{aA}\phi_{bA}. \tag{2.29}$$

Similarily, the affine connection $\Gamma^a_{bc}$ and the torsion tensor $T^a_{bc}$ of the space $\mathcal{M}$ are given by

$$\Gamma^a_{bc} = \mathbf{g}^a \cdot \frac{\partial \mathbf{g}_b}{\partial y^c} = \phi^a_A \partial_b \phi_{cA} \tag{2.30}$$

$$T^a_{bc} = \Gamma^a_{[bc]}$$

respectively. Obviously, the torsion tensor $T^a_{bc}$ is antisymmetric in the indices $c$ and

$b$. We may also prove from (2.28) that the corresponding Riemann-Christoffel tensor is identically zero also, that is,

$$R^{d}_{abc} = 2\partial_{[a}\Gamma^{d}_{b]c} - T^{e}_{ac}\Gamma^{d}_{be} + \Gamma^{d}_{ae}\Gamma^{e}_{bc} \equiv 0. \tag{2.31}$$

It means that the Nonriemannian space $\mathcal{M}$ is flat also.

In the Eulerian description, the dislocation density of the second order can be defined as

$$\alpha^{ab} = \epsilon^{ade}_{(e)} T^{b}_{de} = \epsilon^{ade}_{(e)} \phi^{b}_{A} \partial_{[d}\phi_{e]A} \tag{2.32}$$

where $\epsilon^{abc}_{(e)}$ is the permutation symbol in the $y^{a}$-coordinate system. Obviously the dislocation densities $\alpha^{ab}$ and $\alpha^{\mu\nu}$ are related to each other by the coordinate transformations (2.2a) or (2.2b), i.e.

$$\alpha^{ab} = y^{a}_{\mu} y^{b}_{\nu} \alpha^{\mu\nu}, \qquad \alpha^{\mu\nu} = x^{\mu}_{a} x^{\nu}_{b} \alpha^{ab}. \tag{2.33}$$

## 3. NOETHER'S THEOREM AND EQUATIONS OF VARIATIONAL INVARIANCE

### 3.1 *Noether's theorem*

Let us consider the following action integral

$$I(\Psi^{A_i}) = \int_{E_4} L(z_k; \Psi^{A_i}, \Psi^{A_i}_{k}, \Psi^{A_i}_{kl})\, d^4 z \tag{3.1}$$

taken over a bounded or unbounded region $E_4$ in space. In (3.1), $L$ represents a Lagrangian density depending on the field variables $\Psi^{A_i}$ and their first and second derivatives $\Psi^{A_i}_{k}$ and $\Psi^{A_i}_{kl}$ with respect to $z_k$ in 4-dimensional Euclidean space. We should notice that the dependent variables $\Psi^{A_i}$ $(A_i = A_1, A_2, \ldots)$ with the generalized indices $A_1, A_2, \ldots$, might be scalar, vector, or tensor-valued fields.

As we know, the variational Euler equations of motion following from $\delta I = 0$ in (3.1) for the problem with fixed boundaries are

$$E(L) \equiv \frac{\partial L}{\partial \Psi^{A_i}} - \frac{\partial}{\partial z_k}\left(\frac{\partial L}{\partial \Psi^{A_i}_{k}}\right) + \frac{\partial^2}{\partial z_k \partial z_l}\left(\frac{\partial L}{\partial \Psi^{A_i}_{kl}}\right) = 0 \qquad (k, l = 0, 1, 2, 3). \tag{3.2}$$

For the action integral (3.1), we introduce the small transformations of dependent and independent variables as

$$\bar{z}_k = z_k + \delta z_k \qquad (k = 0, 1, 2, 3) \tag{3.3}$$

and

$$\overline{\Psi}^{A_i}(\bar{z}) = \Psi^{A_i}(z) + \delta\Psi^{A_i} \tag{3.4}$$

where $\delta z_k$ and $\delta\Psi^{A_i}$ represent the variations of $z_k$ and $\Psi^{A_i}$, respectively. With these transformations, the action integral (3.1) changes into

$$\bar{I}(\overline{\Psi}^{A_i}) = \int_{\bar{E}_4} L(\bar{z}_k, \overline{\Psi}^{A_i}, \overline{\Psi}^{A_i}_{k}, \overline{\Psi}^{A_i}_{kl})\, d^4 \bar{z}. \tag{3.5}$$

From (3.3) and (3.4), we may calculate

$$\begin{aligned}
\overline{\Psi}^{A_i}(\bar{z}) &= \Psi^{A_i}(z) + \delta_*\Psi^{A_i} + \Psi^{A_i}_{k}\delta z_k + O(\delta z^2) \\
\overline{\Psi}^{A_i}_{k}(\bar{z}) &= \Psi^{A_i}_{k}(z) + \delta_*\Psi^{A_i}_{k} + \Psi^{A_i}_{kl}\delta z_l + O(\delta z^2) \\
\overline{\Psi}^{A_i}_{kl}(\bar{z}) &= \Psi^{A_i}_{kl}(z) + \delta_*\Psi^{A_i}_{kl} + \Psi^{A_i}_{klm}\delta z_m + O(\delta z^2)
\end{aligned} \tag{3.6}$$

with $\Psi^{A_i}_{klm}$ representing the third derivatives of $\Psi^{A_i}$ with respect to $z_k$ and $\delta_*$ means a variational operator only due to the transformation of field variables $\Psi^{A_i}$ themselves, i.e.

$$\begin{aligned} \delta_* \Psi^{A_i} &= \bar{\Psi}^{A_i}(z) - \Psi^{A_i}(z) \\ \delta_* \Psi^{A_i}_k &= \bar{\Psi}^{A_i}_k(z) - \Psi^{A_i}_k(z) \\ \delta_* \Psi^{A_i}_{kl} &= \bar{\Psi}^{A_i}_{kl}(z) - \Psi^{A_i}_{kl}(z). \end{aligned} \tag{3.7}$$

In addition, the new volume element $d^4\bar{z}$ becomes

$$d^4\bar{z} = [1 + (\delta z_k)_k]\, d^4 z. \tag{3.8}$$

Substituting (3.6) and (3.8) into (3.5) and making use of Taylor series expansion technique, after some algebraic manipulation, we obtain

$$\bar{I}(\bar{\Psi}^{A_i}) = I(\Psi^{A_i}) + \int_{E_4} [E(L)\delta_* \Psi^{A_i} + \nabla_k F_k]\, d^4 z + O(\delta z^2) \tag{3.9}$$

where the operator $E$ on $L$ is given in (3.2), $\nabla_k$ means a 4-D divergence operator and $F_k$ is expressed by

$$F_k = B_{kl}\delta z_l + \left[\frac{\partial L}{\partial \Psi^{A_i}_k} - \frac{\partial}{\partial z_l}\left(\frac{\partial L}{\partial \Psi^{A_i}_{kl}}\right)\right] \delta\Psi^{A_i} + \frac{\partial L}{\partial \Psi^{A_i}_{kl}}\, \delta\Psi^{A_i}_l - \frac{\partial L}{\partial \Psi^{A_i}_{kl}}\, \Psi^{A_i}_{m}(\delta z_m)_l \tag{3.10}$$

with

$$B_{kl} = L\delta_{kl} - \frac{\partial L}{\partial \Psi^{A_i}_k}\, \Psi^{A_i}_l + \nabla_m \left(\frac{\partial L}{\partial \Psi^{A_i}_{km}}\right) \Psi^{A_i}_l - \frac{\partial L}{\partial \Psi^{A_i}_{km}}\, \Psi^{A_i}_{ml}. \tag{3.11}$$

From (3.9), we come to the conclusion: if the fields $\Psi^{A_i}$ ($A_i = A_1, A_2, \ldots$) satisfy the corresponding Euler equations (3.2), then the functional (3.1) remains infinitesimally invariant at $\Psi^{A_i}$ under the small transformations (3.3) and (3.4) if and only if $\Psi^{A_i}$ also satisfies

$$\nabla_k F_k = 0. \tag{3.12}$$

Equation (3.12) which we call the equation of variational invariance, is the mathematical version of the celebrated Noether's theorem.

In what follows, we shall apply this basic formalism (3.12) to derive the conservation laws for the elastic dislocation continuum by means of both Lagrangian and Eulerian representations, and discuss the duality principle of these two sets of conservation laws.

### 3.2 *Variational formulation for a dislocation continuum*

Generally speaking, a material body containing a large number of moving dislocations could not possibly be considered as a conservative system. During the motion of dislocations, the macroscopic plastic deformation takes place as an irreversible thermo-mechanical process and the plastic work done by stresses is irreversibly converted into the thermal energy, which will lead to the evident increase of temperature inside the body. Meanwhile, other irreversible effects, due to heat conduction and viscous dissipation are, in general, involved. In this sense, the conservation laws which are valid for the perfect elastic medium no longer exist for the medium considered here. However, if the deformation which occurs in the material is not large so that all the irreversible effects can be ignored, i.e. the variations of entropy and temperature inside body are not significant, the conservation relations can be established within the framework of elastic dislocation continuum theory, as was done for a perfect elastic medium[18, 19].

In the following derivation, for simplicity, the coordinates $x^\mu$ and $y^a$ are always assumed to be rectilinear. Thus, in Langrangian representation, we assume in (3.1) that

$$z_0 = t, \qquad z_k = x_\mu \qquad (k = \mu = 1, 2, 3) \tag{3.13}$$

and the fields $\Psi^{A_i}$ are given by

$$\Psi^{A_i} \equiv \begin{cases} y_a & A_i = A_1 = a \\ \phi_{\mu A} & A_i = A_2 = \mu A. \end{cases} \tag{3.14}$$

If the Lagrangian function of the system is assumed to depend not only on macroscopic kinetic and elastic strain energies but also on dislocation density, as well as its time derivative, then it takes the general form

$$L = L(x_\mu, t; y_a, \dot{y}_a, y_{a\mu}, \phi_{\mu A}, \dot{\phi}_{\mu A}, \phi_{\mu A\nu}, \dot{\phi}_{\mu A\nu}) \tag{3.15}$$

with the notations

$$\dot{\phi}_{\mu A} = \frac{\partial \phi_{\mu A}}{\partial t} \qquad \phi_{\mu A\nu} = \frac{\partial \phi_{\mu A}}{\partial x_\nu} \qquad \dot{\phi}_{\mu A\nu} = \frac{\partial^2 \phi_{\mu A}}{\partial t \partial x_\nu}. \tag{3.16}$$

The Euler equations of motion corresponding to (3.2) are

$$\begin{aligned} \frac{\partial L}{\partial y_a} - \frac{\partial}{\partial x_\mu}\left[\frac{\partial L}{\partial y_{a\mu}}\right] &= \frac{\partial}{\partial t}\left[\frac{\partial L}{\partial \dot{y}_a}\right] \\ \frac{\partial L}{\partial \phi_{\mu A}} - \frac{\partial}{\partial x_\nu}\left[\frac{\partial L}{\partial \phi_{\mu A\nu}}\right] &= \frac{\partial}{\partial t}\left[\frac{\partial L}{\partial \dot{\phi}_{\mu A}} - \frac{\partial}{\partial x_\nu}\left(\frac{\partial L}{\partial \dot{\phi}_{\mu A\nu}}\right)\right]. \end{aligned} \tag{3.17}$$

For the following small transformations of $x_\mu$, $t$, $y_a$, and $\phi_{\mu A}$

$$\begin{aligned} \bar{x}_\mu &= x_\mu + \delta x_\mu \\ \bar{t} &= t + \delta t \\ \bar{y}_a &= y_a + \delta y_a \\ \bar{\phi}_{\mu A} &= \phi_{\mu A} + \delta\phi_{\mu A} \end{aligned} \tag{3.18}$$

eqn (3.12) is specified in the form

$$\frac{\partial}{\partial t}\left\{e\delta t + b_\mu \delta x_\mu + P_a \delta y_a + S_{\mu A}\delta\phi_{\mu A} + \frac{\partial L}{\partial \dot{\phi}_{\mu A\nu}}\phi_{\lambda A\nu}(\delta x_\lambda)_\mu\right\}$$
$$+ \frac{\partial}{\partial x_\mu}\left\{e_\mu \delta t + b_{\mu\nu}\delta x_\nu + P_{a\mu}\delta y_a + S_{\nu A\mu}\delta\phi_{\nu A} + \frac{\partial L}{\partial \dot{\phi}_{\nu A\mu}}\delta\dot{\phi}_{\nu A}\right\} = 0 \tag{3.19}$$

where the following abbreviations were introduced

$$\begin{aligned} e &\equiv L - \frac{\partial L}{\partial \dot{y}_a}\dot{y}_a + \left[\frac{\partial}{\partial x_\mu}\left(\frac{\partial L}{\partial \dot{\phi}_{\sigma A\mu}}\right) - \frac{\partial L}{\partial \dot{\phi}_{\sigma A}}\right]\dot{\phi}_{\sigma A} - \frac{\partial L}{\partial \dot{\phi}_{\sigma A\mu}}\dot{\phi}_{\sigma A\mu} \\ e_\mu &\equiv -\frac{\partial L}{\partial y_{a\mu}}\dot{y}_a + \left[\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{\phi}_{\sigma A\mu}}\right) - \frac{\partial L}{\partial \phi_{\sigma A\mu}}\right]\dot{\phi}_{\sigma A} - \frac{\partial L}{\partial \dot{\phi}_{\sigma A\mu}}\ddot{\phi}_{\sigma A} \\ b_\mu &\equiv -\frac{\partial L}{\partial \dot{y}_a}y_{a\mu} + \left[\frac{\partial}{\partial x_\nu}\left(\frac{\partial L}{\partial \dot{\phi}_{\sigma A\nu}}\right) - \frac{\partial L}{\partial \dot{\phi}_{\sigma A\nu}}\right]\phi_{\sigma A\mu} - \frac{\partial L}{\partial \dot{\phi}_{\sigma A\nu}}\phi_{\sigma A\nu\mu} \\ b_{\mu\nu} &\equiv L\delta_{\mu\nu} - \frac{\partial L}{\partial y_{a\nu}}y_{a\mu} + \left[\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{\phi}_{\sigma A\nu}}\right) - \frac{\partial L}{\partial \phi_{\sigma A\nu}}\right]\phi_{\sigma A\mu} - \frac{\partial L}{\partial \dot{\phi}_{\sigma A\mu}}\dot{\phi}_{\sigma A\mu} \end{aligned} \tag{3.20a}$$

as well as

$$P_a \equiv \frac{\partial L}{\partial \dot{y}_a} - \frac{\partial}{\partial x_\mu}\left[\frac{\partial L}{\partial \dot{y}_{a\mu}}\right]$$
$$P_{a\mu} \equiv \frac{\partial L}{\partial y_{a\mu}} - \frac{\partial}{\partial t}\left[\frac{\partial L}{\partial \dot{y}_{a\mu}}\right] \tag{3.20b}$$
$$S_{\mu A} \equiv \frac{\partial L}{\partial \dot{\phi}_{\mu A}} - \frac{\partial}{\partial x_\nu}\left[\frac{\partial L}{\partial \dot{\phi}_{\mu A \nu}}\right]$$
$$S_{\nu A\mu} \equiv \frac{\partial L}{\partial \phi_{\nu A\mu}} - \frac{\partial}{\partial t}\left[\frac{\partial L}{\partial \dot{\phi}_{\nu A\mu}}\right].$$

Similarily, in Eulerian representation, we assume in (3.1) that

$$z_0 = t, \qquad z_k = y_a \qquad (k = a = 1, 2, 3) \tag{3.21}$$

and

$$\Phi^{A_i} \equiv \begin{cases} x_\mu & A_i = A_1 = \mu \\ \phi_{aA} & A_i = A_2 = aA. \end{cases} \tag{3.22}$$

Here we have used the notation $\Phi^{A_i}$ to replace $\Psi^{A_i}$ in the Eulerian representation and the Lagrangian density takes the form

$$\mathcal{L} = \mathcal{L}(y_a, t; x_\mu, \dot{x}_\mu, x_{\mu a}, \phi_{aA}, \dot{\phi}_{aA}, \phi_{aAb}, \dot{\phi}_{aAb}). \tag{3.23}$$

The Euler equations of motion corresponding to (3.2) are expressed by

$$\frac{\partial \mathcal{L}}{\partial x_\mu} - \frac{\partial}{\partial y_a}\left[\frac{\partial \mathcal{L}}{\partial x_{\mu a}}\right] = \frac{\partial}{\partial t}\left[\frac{\partial \mathcal{L}}{\partial \dot{x}_\mu}\right]$$
$$\frac{\partial \mathcal{L}}{\partial \phi_{aA}} - \frac{\partial}{\partial y_b}\left[\frac{\partial \mathcal{L}}{\partial \phi_{aAb}}\right] = \frac{\partial}{\partial t}\left[\frac{\partial \mathcal{L}}{\partial \dot{\phi}_{aA}} - \frac{\partial}{\partial y_b}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}_{aAb}}\right)\right]. \tag{3.24}$$

With the small transformations of $y_a$, $t$, $x_\mu$, and $\phi_{aA}$ as given by

$$\bar{y}_a = y_a + \delta y_a$$
$$\bar{t} = t + \delta t \tag{3.25}$$
$$\bar{x}_\mu = x_\mu + \delta x_\mu$$
$$\bar{\phi}_{aA} = \phi_{aA} + \delta\phi_{aA}$$

the eqn (3.12) is also specified in the form

$$\frac{\partial}{\partial t}\left\{E\delta t + p_a \delta y_a + B_\mu \delta x_\mu + S_{aA}\delta\phi_{aA} + \frac{\partial \mathcal{L}}{\partial \dot{\phi}_{aAb}}\phi_{cAb}(\delta y_c)_a\right\}$$
$$+ \frac{\partial}{\partial y_a}\left\{E_a \delta t + p_{ab}\delta y_b + B_{\mu a}\delta x_\mu + S_{bAa}\delta\phi_{bA} + \frac{\partial \mathcal{L}}{\partial \dot{\phi}_{bAa}}\delta\dot{\phi}_{bA}\right\} = 0 \tag{3.26}$$

where we introduced the following abbreviations

$$E \equiv \mathcal{L} - \frac{\partial \mathcal{L}}{\partial \dot{x}_\mu}\dot{x}_\mu + \left[\frac{\partial}{\partial y_a}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}_{cAa}}\right) - \frac{\partial \mathcal{L}}{\partial \dot{\phi}_{cA}}\right]\dot{\phi}_{cA} - \frac{\partial \mathcal{L}}{\partial \dot{\phi}_{cAa}}\dot{\phi}_{cAa}$$

$$
\begin{aligned}
E_a &\equiv \frac{\partial \mathcal{L}}{\partial x_{\mu a}} \dot{x}_\mu + \left[ \frac{\partial}{\partial t} \left( \frac{\partial \mathcal{L}}{\partial \dot{\phi}_{,Aa}} \right) - \frac{\partial \mathcal{L}}{\partial \phi_{,Aa}} \right] \dot{\phi}_{,A} - \frac{\partial \mathcal{L}}{\partial \dot{\phi}_{,Aa}} \ddot{\phi}_{,A} \\
p_a &\equiv - \frac{\partial \mathcal{L}}{\partial \dot{x}_\mu} x_{\mu a} + \left[ \frac{\partial}{\partial y_b} \left( \frac{\partial \mathcal{L}}{\partial \dot{\phi}_{,Ab}} \right) - \frac{\partial \mathcal{L}}{\partial \dot{\phi}_{,A}} \right] \phi_{,Aa} - \frac{\partial \mathcal{L}}{\partial \dot{\phi}_{,Ab}} \phi_{,Aba} \\
p_{ab} &\equiv \mathcal{L}\delta_{ab} - \frac{\partial \mathcal{L}}{\partial x_{\mu b}} x_{\mu a} + \left[ \frac{\partial}{\partial t} \left( \frac{\partial \mathcal{L}}{\partial \dot{\phi}_{,Ab}} \right) - \frac{\partial \mathcal{L}}{\partial \phi_{,Ab}} \right] \phi_{,Aa} - \frac{\partial \mathcal{L}}{\partial \dot{\phi}_{,Ab}} \dot{\phi}_{,Aa}
\end{aligned}
\tag{3.27a}
$$

as well as

$$
\begin{aligned}
B_\mu &\equiv \frac{\partial \mathcal{L}}{\partial \dot{x}_\mu} - \frac{\partial}{\partial y_a} \left[ \frac{\partial \mathcal{L}}{\partial \dot{x}_{\mu a}} \right] \\
B_{\mu a} &\equiv \frac{\partial \mathcal{L}}{\partial x_{\mu a}} - \frac{\partial}{\partial t} \left[ \frac{\partial \mathcal{L}}{\partial \dot{x}_{\mu a}} \right] \\
S_{aA} &\equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}_{aA}} - \frac{\partial}{\partial y_b} \left[ \frac{\partial \mathcal{L}}{\partial \dot{\phi}_{aAb}} \right] \\
S_{aAb} &\equiv \frac{\partial \mathcal{L}}{\partial \phi_{aAb}} - \frac{\partial}{\partial t} \left[ \frac{\partial \mathcal{L}}{\partial \dot{\phi}_{aAb}} \right] .
\end{aligned}
\tag{3.27b}
$$

In comparing (3.19) to (3.26), we observe that these two equations of variational invariance are dual in form, where the role of $x_\mu$ and $y_a$ is merely interchanged. Using these two basic equations, the conservation laws of various kinds can be derived using the duality principle of conservation laws as illustrated below.

## 4. CONSERVATION LAWS AND THEIR DUALITY PRINCIPLE

A. G. Herrmann[13, 19] discussed in some detail the conservation laws and their principle of duality in elastic continua in terms of an alternate and simpler procedure. The results have been extended to finite elastic medium by Duan and G. Herrmann[19]. They showed that if the same kind of specific transformation of either independent or dependent variables which can keep both the action integrals infinitesimally invariant is applied simultaneously to the basic Lagrangian and Eulerian conservation equations (3.19) and (3.26), one can obtain the corresponding specific conservation laws. These laws are expressed in different mathematical forms but contain the same physical information, therefore they are dual. Now, following this rule and using the Cartesian coordinates for $x_\mu$ and $y_a$, we shall discuss the duality principle of conservation laws in a dislocation continuum in terms of vielbein theory.

*Time translation and energy conservation law*

The time translation can be expressed in the form

$$\delta t = \epsilon_t, \qquad \delta x_\mu = 0, \qquad \delta y_a = \delta\phi_{\mu A} = 0 \tag{4.1}$$

in Lagrangian representation and

$$\delta t = \epsilon_t, \qquad \delta y_a = 0, \qquad \delta x_\mu = \delta\phi_{aA} = 0 \tag{4.2}$$

in Eulerian representation, where $\epsilon_t$ is a small time parameter. Substituting (4.1) and (4.2) into (3.19) and (3.26) respectively, we obtain

$$\frac{\partial e}{\partial t} + \frac{\partial e_\mu}{\partial x_\mu} = 0 \tag{4.3a}$$

and

$$\frac{\partial E}{\partial t} + \frac{\partial E_a}{\partial y_a} = 0. \tag{4.3b}$$

These two equations are known as the dual forms of the energy conservation laws, which hold true if both Lagrangians $L$ and $\mathcal{L}$ do not depend on $t$ explicitly.

*Translation of material coordinate frame and material momentum conservation law*

Let

$$\delta t = 0, \qquad \delta x_\mu = \epsilon_\mu \tag{4.4}$$

in (3.19) and (3.26), where $\epsilon_\mu$ are three small parameters of the same order. Since the translation of the coordinate $x_\mu$ does not make any changes in $y_a$, and $\phi_{\mu A}$, we have

$$\delta y_a = \delta\phi_{\mu A} = 0 \tag{4.5}$$

in (3.19) and

$$\delta y_a = \delta\phi_{aA} = 0 \tag{4.6}$$

in (3.26). Substituting (4.4) and (4.5) into (3.19) and (3.26) respectively, we obtain

$$\frac{\partial b_\mu}{\partial t} + \frac{\partial b_{\mu\nu}}{\partial x_\nu} = 0. \tag{4.7a}$$

Similarily,

$$\frac{\partial B_\mu}{\partial t} + \frac{\partial B_{\mu a}}{\partial y_a} = 0. \tag{4.7b}$$

These two equations respresent the material momentum conservation laws and are dual to each other. As shown for a perfect elastic continuum, these conservation laws play an important role in elastic fracture mechanics in determining J-integrals and they hold true if and only if the Lagrangians $L$ and $\mathcal{L}$ do not depend on $x^\mu$ explicitly.

*Translation of spatial coordinates frame and physical momentum conservation law*

In classical physics, it is known that the translation of space coordinates $y_a$ leads to linear momentum conservation laws. This also holds true in dislocation continuum mechanics. To show this, let us assume

$$\delta t = 0, \qquad \delta y_a = \epsilon_a \tag{4.8}$$

in (3.19) and (3.26), respectively, where $\epsilon_a$ are three small parameters. For the same reason as presented above, translation of spatial coordinate frame does not make any change in $x_\mu$ and $\phi_{\mu A}$ or $\phi_{aA}$, therefore, we have from (3.19) and (3.26) that

$$\delta x_\mu = 0, \qquad \delta\phi_{\mu A} = 0 \tag{4.9}$$

and

$$\delta x_\mu = 0, \qquad \delta\phi_{aA} = 0. \tag{4.10}$$

Substitution of (4.8) and (4.9) into (3.19) leads to the following physical momentum conservation law in Lagrangian representation,

$$\frac{\partial P_a}{\partial t} + \frac{\partial P_{a\mu}}{\partial x_\mu} = 0. \tag{4.11a}$$

Similarily, we obtain

$$\frac{\partial p_a}{\partial t} + \frac{\partial p_{ab}}{\partial y_b} = 0, \tag{4.11b}$$

which represents the physical momentum conservation law in Eulerian representation. In fact, equations (4.11*a*) and (4.11*b*) are the dual forms of the physical momentum conservation laws and hold true if and only if the Lagrangians $L$ and $\mathcal{L}$ do not depend on $y_a$ explicitly.

If we compare the expressions for $b_\mu$ and $b_{\mu\nu}$ in $(3.20)_{3-4}$ with the expressions for $p_a$ and $p_{ab}$ in $(3.27)_{3-4}$, we may see that $b_\mu$ and $b_{\mu\nu}$ are related to $x_\mu$ and $\phi_{aA}$ in the same fashion as $p_a$ and $p_{ab}$ are related to $y_a$ and $\phi_{\mu A}$. This comparison is also confirmed for $B_\mu$, $B_{\mu a}$ and $P_a$, $P_{a\mu}$. We call $b_\mu$, $b_{\mu\nu}$ (or $B_\mu$, $B_{\mu a}$) the material momenta and $p_a$, $p_{ab}$ (or $P_a$, $P_{a\mu}$) the physical momenta. The material momenta are independent of the physical momenta, therefore, the conservation laws (4.7*a*,*b*) by no means imply the conservation equations (4.11*a*,*b*) and vice versa.

*Rotation of material coordinate frame and material angular momentum conservation laws*

The small rotation of a material coordinate frame can be expressed by

$$\delta x_\mu = \epsilon_{\mu\nu\lambda}\,\alpha_\nu\, x_\lambda, \tag{4.15}$$

where $\epsilon_{\mu\nu\lambda}$ is the permutation symbol, and $\alpha_\nu$ are the three small arbitrarily chosen parameters. If we introduce the notation

$$\delta x_{\mu\lambda} \equiv \frac{\partial(\delta x_\mu)}{\partial x_\lambda} = \epsilon_{\mu\nu\lambda}\,\alpha_\nu \tag{4.16}$$

by which any physical quantity $f_{\mu\nu\ldots}$ having indices $\mu$, $\nu$, . . . is transformed to

$$\delta f_{\mu\nu\ldots} = \delta x_{\mu\lambda} f_{\lambda\nu\ldots} + \delta x_{\nu\lambda} f_{\mu\lambda\ldots} + \ldots. \tag{4.17a}$$

Applying the rule (4.17a) to the vielbein transformations $\delta\phi_{\mu A}$, we obtain

$$\delta\phi_{\mu A} = \delta x_{\mu\nu}\phi_{\nu A}, \qquad \delta\dot{\phi}_{\mu A} = \delta x_{\mu\nu}\dot{\phi}_{\nu A}. \tag{4.17b}$$

Substituting (4.17a,b) into (3.19), we obtain

$$\epsilon_{\mu\nu\lambda}\left(\frac{\partial m_{\mu\nu}}{\partial t} + \frac{\partial m_{\sigma\mu\nu}}{\partial x_\sigma}\right) = 0, \tag{4.18a}$$

where the notation $m_{\mu\nu}$ and $m_{\sigma\mu\nu}$

$$m_{\mu\nu} = b_\mu x_\nu + s_{\mu A}\phi_{\nu A} + \frac{\partial L}{\partial\dot{\phi}_{\nu A\lambda}}\,\phi_{\mu A\lambda} \tag{4.19}$$

and

$$m_{\sigma\mu\nu} = b_{\sigma\mu} x_\nu + s_{\mu A\sigma}\phi_{\nu A} + \frac{\partial L}{\partial\dot{\phi}_{\mu A\sigma}}\,\dot{\phi}_{\nu A} \tag{4.20}$$

represent the material angular momentum.

In Eulerian representation, $x_\mu$ are treated as dependent variables, therefore, the rotation of coordinates $x_\mu$ cannot make any change in the vielbein $\phi_{aA}$ and $y_a$. Substitution of (4.15) into (3.25) leads to

$$\epsilon_{\mu\nu\lambda}\left(\frac{\partial M_{\mu\nu}}{\partial t}+\frac{\partial M_{\mu\nu a}}{\partial y_a}\right)=0, \tag{4.18b}$$

where

$$M_{\mu\nu}=B_\mu x_\nu, \qquad M_{\mu\nu a}=B_{\mu a}x_\nu. \tag{4.21}$$

Equations (4.18*a*) and (4.18*b*) represent the dual forms of material angular momentum conservation law. Equations (4.18*a*) take on a complicated form, but (4.18*b*) is rather simple.

*Rotation of space coordinate frame and conservation laws of physical angular momentum*

The physical moment of momentum conservation law can be derived in the same way as we derived (4.18*a*,*b*) by applying the small rotation transformation of space coordinates $y_a$

$$\delta y_a=\epsilon_{abc}\alpha_b y_c \tag{4.22}$$

to the eqns (3.19) and (3.26). This conservation law is expressed either by

$$\epsilon_{abc}\left(\frac{\partial m_{ab}}{\partial t}+\frac{\partial m_{\mu ab}}{\partial x_\mu}\right)=0 \tag{4.23a}$$

in the Lagrangian representation or by

$$\epsilon_{abc}\left(\frac{\partial M_{ab}}{\partial t}+\frac{\partial M_{dab}}{\partial y_d}\right)=0 \tag{4.23b}$$

in the Eulerian representation, where

$$m_{ab}=P_a y_b, \qquad m_{\mu ab}=P_{a\mu}y_b \tag{4.24}$$

and

$$M_{ab}=p_a y_b+S_{aA}\phi_{bA}++\frac{\partial \mathcal{L}}{\partial \dot{\phi}_{bAe}}\phi_{aAe} \tag{4.25}$$

$$M_{dab}=p_{da}y_b+S_{aAd}\phi_{bA}+\frac{\partial \mathcal{L}}{\partial \dot{\phi}_{aAd}}\dot{\phi}_{bA} \tag{4.26}$$

*Rotation of local anholonomic coordinate frame and gauge angular momentum*

In dealing with the conservation laws for a dislocation continuum, a question arises: except for the conservation laws derived above, does there exist another kind of conservation law which is related to local coordinate transformation of the material body in the n-state? As we stated above, each relaxed volume element in the n-state can rotate freely without leading to any change of the Lagrangian of the system. Therefore, instead of the rotation of the relaxed element, we may also introduce a small rotation transformation of local anholonomic coordinate frame as

$$\bar{\delta} z_A=\epsilon_{ABC}\alpha_B\delta z_C, \tag{4.27}$$

where $\alpha_C$ are small parameters. This rotation should not lead to any change of the Lagrangian of the system. Using the same rule as given in (4.17a), the variations of vielbein and its derivatives are given by

$$\begin{aligned}
\bar{\delta}\phi_{\mu A} &= \epsilon_{ABC}\alpha_B\phi_{\mu C} \\
\bar{\delta}\dot{\phi}_{\mu A} &= \epsilon_{ABC}\alpha_B\dot{\phi}_{\mu C} \\
\bar{\delta}\phi_{\mu A\nu} &= \epsilon_{ABC}\alpha_B\phi_{\mu C\nu} \\
\bar{\delta}\dot{\phi}_{\mu A\nu} &= \epsilon_{ABC}\alpha_B\dot{\phi}_{\mu C\nu}.
\end{aligned} \tag{4.28}$$

Since the rotation of the local anholonomic coordinate system is independent of the coordinates $x_\mu$ or $y_a$ and does not change in the indices $\mu$ and $a$, under this transformation (4.27), we have, therefore,

$$\delta t = \delta x_\mu = \delta y_a = 0. \tag{4.29}$$

Substituting (4.28) and (4.29) into (3.19), we find the following conservation equations

$$\epsilon_{AEC}\left[\frac{\partial}{\partial t}(S_{\mu A}\phi_{\mu C}) + \frac{\partial}{\partial x_\nu}\left(S_{\mu A\nu}\phi_{\mu C} + \frac{\partial L}{\partial \dot{\phi}_{\mu A\nu}}\dot{\phi}_{\mu C}\right)\right] = 0. \tag{4.30a}$$

In a very similar way, under the transformation (4.27), eqns (3.26) reduce to

$$\epsilon_{AEC}\left[\frac{\partial}{\partial t}(S_{aA}\phi_{aC}) + \frac{\partial}{\partial y_a}\left(S_{bAa}\phi_{bC} + \frac{\partial \mathcal{L}}{\partial \dot{\phi}_{bAa}}\dot{\phi}_{bC}\right)\right] = 0 \tag{4.30b}$$

in the Eulerian representation.

Finally, we should notice here that since the transformation (4.27) is independent of time and point coordinates, regardless of whether or not the Lagrangian function $L$ or $\mathcal{L}$ depends on time or the coordinates explicitly, the "gauge angular momentum" conservation laws (4.30*a*) and (4.31*b*) always hold true. In a separate study, we shall further discuss the physical meaning of this conservation law.

## 5. CONCLUDING REMARKS

Combining the vielbein theory of dislocations with Noether's theorem, an effective method is presented to deal with conservation laws and their duality principles in such media. Besides the conventional conservation laws derived from the time translation and translation and rotation of material and spatial coordinate frames, the procedure yields an additional conservation law termed the "gauge angular momentum" conservation law by employing a small rotation of the local coordinate frame to the variational invariance equations. When there are no dislocations inside the body, all conservation laws derived above will reduce to those studied extensively in elastic (linear or nonlinear) continua[15–18]. In other words, the results given in the study can be considered as a natural extension of the elastic continuum theory to dislocation continuum theory.

All conservation laws are expressed in 4-dimensional divergence-free forms. For the static problem, the conservation laws can be represented through so-called path-independent integral forms which are of major importance in the study of defect and fracture mechanics. In this study, we did not mention any specific forms of the Lagrangian functions. We would like to point out here, however, that any physical quantities appearing in the Lagrangian must remain not only covariant with respect to coordinate transformations but invariant with respect to local coordinate transformations as well. Following this principle, the elastic strain tensor, dislocation density tensors and their time differentials are suggested to be such proper quantities. Once the Lagrangian of a dislocation continuum system is given in a specific form, all the above derived conservation laws can be written in their specific form. In a separate study, we shall discuss this issue in some detail, where special attention will be given to the problem of determining the dependence of the Lagrangian on its determining parameters and the practical application of path-independent integrals based on conservation laws.

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## REFERENCES

1. K. Kondo, *Proc. Jap. Nat. Cing. Appl. Mech* **40** (1952).
2. K. Kondo, *RAAG Memoirs 1-4*, Dir. D. Gakajutsu Bunken Fukyukai, Tokoy (1955, 1958, 1962, 1968).
3. E. Kröner and R. Rieder, *Zeit. der Physik* **145**, 424 (1956).
4. E. Kröner, *Kontinuum Theorie der Verz. und Eigenspann.*, Springer-Verlag, Berlin (1958).
5. B. A. Bilby *et al.*, *Proc. R. Soc. London* **A321**, 263 (1955).
6. B. A. Bilby, *Progress in Solid Mechanics* (Edited by I. N. Sneddon and R. Hill), Vol. 1, 331, North-Holland, Amsterdam, (1960).
7. L. I. Sedov, *UMN* **26**, 125 (1956).
8. L. I. Sedov and V. L. Berditchevski, Mech. of generalized continua, (Edited by E. Kröner) *IUTAM symposium*, 214 (1967).
9. F. Bloom, *Lecture Notes in Mathematics*, Vol. 733, Springer-Verlag, Berlin (1979).
10. S. Amari, *RAAG Memoirs 3*, D-XV (1968).
11. E. Kröner, Mechanics of Generalized Continua, IUTAM symposium, Springer-Verlag, N.Y. Inc. (1968).
12. J. D. Eshelby, The force on an elastic singularity. *Phil. Tran. Roy. Soc. London* **A244**, 87 (1951).
13. A. Golebiewska Herrmann, On conservation laws of continuum mechanics. *Int. J. Solids Structures* **17**, 1 (1981).
14. Y. S. Duan and Z. P. Duan, *Gauge Field Theory of Dislocation and Disclination Continuum*, SLAC-PUB-286, (1984).
15. W. Günther, Uber einige randintegral der elastomechanik. *Abh. Braunschw. Wiss. Ges.* **14**, 54 (1962).
16. J. K. Knowles and Eli Sternberg, On a class of conservation laws in linearized and finite elastostatics, *Arch. Rat. Mech. Anal.* **44**, 187 (1972).
17. D. C. Fletcher, Conservation laws in linear elastodynamics. *Arch. Rat. Mech. Anal.* **60**, 329 (1976).
18. Noether, E., "Invariant variational problems," *Trans. Th. Stat. Phys.* (trans. Tavel, M.), **1**, 186 (1971). Also see Nachr. Ges. Göttingen (Math-Phys. Klasse), Vol. 3, 235 (1918).
19. Z. P. Duan and G. Herrmann, On the duality principle of conservation laws in finite elastodynamics. In press.
20. A. Golebiewska Herrmann, *Int. J. Engng Sci.* **16**, 329 (1978).
21. C. Møller, *The Theory of Relativity*, Chap. XI, Sec. 126 (1955).
22. D. R. Brill and J. A. Wheeler, *Rev. of Mod. Phy.* **29**, 465 (1957).
23. Y. S. Duan and J. Y. Zhang, *Acta Physica Sinica* **19**, 689 (1963).
24. A. C. Eringen *et al. Continuum Physicas*, Vol. II, Academic Press, New York (1975).
25. A. C. Eringen and E. S. Suhubi, *Elastodynamics*, Vol. I. Finite Motion, Academic Press, New York (1974).
26. B. K. D. Gairola, Nonlinear elastic problem, in *Dislocation in Solids* (Edited by F. R. N. Nabarro), North-Holland Publishing Company, (1979).